

Chapter 8 Approximation theory

§ Discrete least square approximation

A. Given $y = ax + b$ and $\{x_i, f(x_i)\}_{i=1}^{i=n}$ to $\min \sum_{i=1}^n [y_i - f(x_i)]^2$

$$\frac{\partial}{\partial a} \sum_{i=1}^n [y_i - f(x_i)]^2 = 0, \quad \frac{\partial}{\partial b} \sum_{i=1}^n [y_i - f(x_i)]^2 = 0 \quad \text{imply}$$

$$a \left(\sum_{i=1}^n x_i^2 \right) + b \left(\sum_{i=1}^n x_i \right) = \sum_{i=1}^n x_i f(x_i), \quad a \left(\sum_{i=1}^n x_i \right) + b \left(\sum_{i=1}^n 1 \right) = \sum_{i=1}^n f(x_i)$$

or

$$\begin{bmatrix} \sum_{i=1}^n x_i & \sum_{i=1}^n 1 \\ \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^n f(x_i) \\ \sum_{i=1}^n x_i f(x_i) \end{Bmatrix}$$

B. Given $y = ax^2 + bx + c$ and $\{x_i, f(x_i)\}_{i=1}^{i=n}$ to $\min \sum_{i=1}^n [y_i - f(x_i)]^2$

$$\frac{\partial}{\partial a} \sum_{i=1}^n [y_i - f(x_i)]^2 = 0, \quad \frac{\partial}{\partial b} \sum_{i=1}^n [y_i - f(x_i)]^2 = 0, \quad \frac{\partial}{\partial c} \sum_{i=1}^n [y_i - f(x_i)]^2 = 0$$

imply

$$\begin{bmatrix} \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i & \sum_{i=1}^n 1 \\ \sum_{i=1}^n x_i^3 & \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i^4 & \sum_{i=1}^n x_i^3 & \sum_{i=1}^n x_i^2 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^n f(x_i) \\ \sum_{i=1}^n x_i f(x_i) \\ \sum_{i=1}^n x_i^2 f(x_i) \end{Bmatrix}$$

§ Continuous least square approximation

A. Given $f \in C[a, b]$ to find $P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

Such that $E(a_0, a_1, \dots, a_n) = \int_a^b [f(x) - a_n x^n - \dots - a_0]^2 dx$ to be minimum.

$$\frac{\partial}{\partial a_i} E(a_0, \dots, a_n) = 0 \quad \text{implies} \quad \int_a^b [f(x) - a_n x^n - \dots - a_0] x^i dx = 0$$

$$\begin{bmatrix} \int_a^b 1 dx & \bullet & \bullet & \int_a^b x^n dx \\ \bullet & \int_a^b x^2 dx & \bullet & \int_a^b x^{n+1} dx \\ \bullet & \bullet & \bullet & \bullet \\ \int_a^b x^n dx & \int_a^b x^{n+1} dx & \bullet & \int_a^b x^{2n} dx \end{bmatrix} \begin{Bmatrix} a_0 \\ \bullet \\ \bullet \\ a_n \end{Bmatrix} = \begin{Bmatrix} \int_a^b f(x) dx \\ \bullet \\ \bullet \\ \int_a^b x^n f(x) dx \end{Bmatrix}$$

B. Minimum

$$E(a_0, a_1, \dots, a_n) = \int_a^b w[f(x) - P(x)]^2 dx$$

where $w(x)$ is a weighting function and

$$P(x) = \sum_{i=1}^n a_i \phi_i(x) \quad \text{with} \quad \int_a^b w \phi_i \phi_j dx = \begin{cases} 0 & i \neq j \\ \alpha_i & i = j \end{cases}$$

The solution is obtained from $\frac{\partial}{\partial a_i} E(a_0, a_1, \dots, a_n) = 0$ is

$$\int_a^b w f(x) dx = \int_a^b \sum_{i=1}^n w a_i \phi_i \phi_j dx = a_i \int_a^b w \phi_i^2 dx$$

$$\text{or } a_i = \int_a^b w f(x) dx / \int_a^b w \phi_i^2 dx$$

Ex. Generate a set of polynomial $\{\phi_0, \dots\}$ such that $\int_a^b w \phi_i \phi_j dx = 0$

for $i \neq j$.

$$\phi_0(x) = 1, \quad \phi_1(x) = x - B_1, \quad \phi_k(x) = (x - B_k) \phi_{k-1}(x) - C_k \phi_{k-2}(x), \quad k \geq 2.$$

$$\int_a^b w \times 1 \times (x - B_1) dx = 0 \quad \text{implies} \quad B_1 = \int_a^b w \times 1 \times x dx / \int_a^b w \times 1 dx,$$

$$\int_a^b w \phi_k \phi_{k-1} dx = 0 \quad \text{implies} \quad \int_a^b w \phi_{k-1} [(x - B_k) \phi_{k-1} - C_k \phi_{k-2}] dx = 0$$

$$\text{or } B_k = \int_a^b w x \phi_{k-1}^2 dx / \int_a^b w \phi_{k-1}^2 dx$$

$$\int_a^b w \phi_k \phi_{k-2} dx = 0 \quad \text{implies} \quad \int_a^b w \phi_{k-2} [(x - B_k) \phi_{k-1} - C_k \phi_{k-2}] dx = 0$$

$$\text{or } C_k = \int_a^b w x \phi_{k-2} \phi_{k-1} dx / \int_a^b w \phi_{k-2}^2 dx$$

Ex. Legendre polynomials $w(x) = 1$

$$p_0(x) = 1, \quad B_1 = 0, \quad p_1(x) = x, \quad B_2 = 0, \quad C_2 = \frac{1}{3}, \quad p_2(x) = x^2 - \frac{1}{3},$$

$$p_3(x) = x^3 - \frac{3}{5}x, \quad \dots, \quad p_{n+1}(x) = x p_n(x) - \frac{n^2}{4n^2 - 1} p_{n-1}(x)$$

Ex. Chebyshev polynomials

$$T_n(x) = \cos(n \cos^{-1} x), \quad n \geq 0, \quad x \in [-1, 1]$$

Let $\theta = \cos^{-1} x$, $T_n(x) = \cos(n\theta) = T_n(\cos \theta)$, $\theta \in [0, \pi]$,

$$T_0(x) = 1, \quad T_1(x) = x,$$

$$T_{n+1}(x) = \cos[(n+1)\theta] = \cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta$$

$$T_{n-1}(x) = \cos[(n-1)\theta] = \cos(n\theta)\cos\theta + \sin(n\theta)\sin\theta$$

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x) \quad \text{or} \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

Ex. Fourier series

$$\{\phi_0, \phi_1, \dots, \phi_{2n-1}\}, \quad \phi_0(x) = \frac{1}{2}, \quad \phi_k(x) = \cos(kx), \quad k = 1 \sim n,$$

$$\phi_{n+k}(x) = \sin(kx), \quad k = 1 \sim n-1, \quad x \in [-\pi, \pi)$$

$$s_n = \frac{1}{2}a_0 + \sum_{k=1}^{2n-1} a_k \phi_k(x), \quad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \phi_k dx$$

Ex. Fast Fourier series

$$\text{Use } S_n(x) = \frac{1}{2}a_0 + a_n \cos(nx) + \sum_{k=1}^{n-1} [a_k \cos kx + b_k \sin kx], \quad x \in [-\pi, \pi)$$

to least square approximate to $\{x_i, y_i\}_{i=0}^{2m-1}$, $x_i = -\pi + (\frac{i}{m})\pi$

$$\text{Error } E(a_0, a_1, \dots, a_n, b_1, \dots, b_{n-1}) = \sum_{i=0}^{2m-1} [y_i - S_n(x_i)]^2$$

$$0 = \frac{\partial E}{\partial a_0} = \sum_{i=0}^{2m-1} [y_i - \frac{1}{2}a_0 - a_n \cos nx_i - \sum_{k=1}^{n-1} (a_k \cos kx_i + b_k \sin kx_i)]$$

$$0 = \frac{\partial E}{\partial a_l} = \sum_{i=0}^{2m-1} [y_i - \frac{1}{2}a_0 - a_n \cos nx_i - \sum_{k=1}^{n-1} (a_k \cos kx_i + b_k \sin kx_i)] \cos lx_i$$

$$0 = \frac{\partial E}{\partial b_l} = \sum_{i=0}^{2m-1} [y_i - \frac{1}{2}a_0 - a_n \cos nx_i - \sum_{k=1}^{n-1} (a_k \cos kx_i + b_k \sin kx_i)] \sin lx_i.$$

$$e^{jkx} = \cos kx + j \sin kx, \quad j = \sqrt{-1}$$

$$\sum_{i=0}^{2m-1} (\cos kx_i + j \sin kx_i) = \sum_{i=0}^{2m-1} e^{jkx_i}, \quad x_i = -\pi + (\frac{i}{m})\pi$$

$$\sum_{i=0}^{2m-1} e^{jkx_i} = \sum_{i=0}^{2m-1} e^{jk[-\pi + \frac{i}{m}\pi]} = e^{-jk\pi} \sum_{i=0}^{2m-1} e^{jk[\frac{i}{m}\pi]} = e^{-jk\pi} \times 0 = 0$$

$$\text{So, } \sum_{j=1}^{2m-1} (\cos kx_j, \sin kx_j, 1) = (0, 0, 2m),$$

$$\cos kx_i \cos lx_i = \frac{1}{2} [\cos(k+l)x_i + \cos(k-l)x_i], \quad \sum_{i=0}^{2m-1} (\cos kx_i \cos lx_i) = \begin{cases} 0 & k \neq l \\ m & k = l \end{cases}$$

$$\sin kx_i \sin lx_i = \frac{1}{2} [\cos(k-l)x_i - \cos(k+l)x_i], \quad \sum_{i=0}^{2m-1} (\sin kx_i \sin lx_i) = \begin{cases} 0 & k \neq l \\ m & k = l \end{cases}$$

$$\cos kx_i \sin lx_i = \frac{1}{2} [\sin(k+l)x_i + \sin(l-k)x_i], \quad \sum_{i=0}^{2m-1} (\cos kx_i \sin lx_i) = 0$$

$$a_0 = \frac{1}{m} \sum_{i=0}^{2m-1} y_i, \quad a_l = \frac{1}{m} \sum_{i=0}^{2m-1} y_i \cos lx_i, \quad b_l = \frac{1}{m} \sum_{i=0}^{2m-1} y_i \sin lx_i$$

Ex. Use $S_n(x) = \frac{1}{2}a_0 + a_n \cos(nx) + \sum_{k=1}^{n-1} [a_k \cos kx + b_k \sin kx], \quad x \in [-\pi, \pi]$

to least square approximate to $\{x_i, y_i\}_{i=0}^{2m-1}$,

$$x_0 = c, \quad x_{2m-1} = d, \quad \Delta x = \frac{d-c}{2m-1}, \quad x_i = x_0 + i\Delta x$$

Transform the interval $[c, d]$ to $[-\pi, \pi]$

Let $z_i = \pi \left[2 \frac{(x_i - c)}{d - c} - 1 \right]$

Then the function $S_n(z) = \frac{1}{2}a_0 + a_n \cos(nz) + \sum_{k=1}^{n-1} [a_k \cos kz + b_k \sin kz], \quad z \in [-\pi, \pi]$

will least square approximate to $\{z_i, y_i\}_{i=0}^{2m-1}$, $z_i = -\pi + \left(\frac{i}{m}\right)\pi$.

HW8

1. Given the data as listed below

x	4.0	4.2	4.5	4.7	5.1	5.5	5.9	6.3
y	102.6	113.2	130.1	142.1	167.5	195.1	224.9	256.8

- Construct the least squares approximation of degree two and compute the error.
 - Construct the least squares approximation of the form be^{ax} and compute the error.
 - Construct the least squares approximation of the form bx^n and compute the error.
2. Find the least squares polynomial approximation of degree two on the interval $[-1, 1]$ for the function $f(x) = \frac{1}{2}\cos x + \frac{1}{4}\sin 2x$
3. Determine the discrete least squares trigonometric polynomial S_4

using $m = 16$ for $f(x) = x^2 \sin x$ on the interval $[0, 1]$.

b. Compute $\int_0^1 S_4(x) dx$

c. Compare the integral in part (b) to $\int_0^1 x^2 \sin x dx$

d. Compute the error $E(S_4)$